

# Bell inequalities for a sensible family of local hidden variable theories testable at low detection efficiency

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Received 27 December 2006 / Received in final form 7 March 2007

Published online 23 March 2007 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2007

**Abstract.** A family of local models containing two angles as hidden variables is defined for experiments measuring polarization correlation of optical photons. Searching for the best model of the family, that is giving predictions most close to quantum mechanics, allows deriving Bell-type inequalities which may be tested with relatively low detection efficiency.

**PACS.** 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 42.50.Ar Photon statistics and coherence theory

## 1 Introduction

More than forty years have elapsed since John Bell [1] proposed his celebrated inequalities. These inequalities, which involve measurable quantities, provide necessary conditions for local realism and, in some experiments with ideal set-ups, contradict the predictions of quantum mechanics. Many empirical tests have been performed of local realism against quantum mechanics, via the Bell inequalities, but no experiment has been conclusive. In fact although the results have generally agreed with the predictions of quantum mechanics, no experiment has given results incompatible with local realism, as is shown by the existence of local realistic models for all of them [2]. The inability to perform a true empirical test of local realism is commonly disguised with the claim that it has already been refuted by the experiments, modulo some irrelevant loopholes. But I think that the extreme difficulty to make a loophole-free test, proved by the unsuccessful effort of forty years, does not support the common wisdom that the question of local realism is settled. On the contrary the conclusion is that further research is needed.

Locality should be understood in the relativistic sense, that is impossibility of superluminal communication. Thus testing locality requires measurements made in regions *spatially* separated, in the sense of relativity theory. As a consequence the tests are extremely difficult with massive particles [3] and reliable experiments must be performed with photons. On the other hand, tests with high energy photons are not possible due to the lack of efficient polarization analyzers. These difficulties have caused that most of the experimental tests have been performed with optical photons, where good polarization analyzers exist and

locality may be insured. However these experiments suffer from a detection loophole, due to the fact that a good overall detection efficiency has not yet been achieved. In fact, it is well-known that efficiencies as high as 80% are required for loophole-free tests of local realism. Detectors with high quantum efficiency already exist, but there are other difficulties reducing the overall efficiency to about 30% or less in practice.

It is common to formulate the question of local realism in terms of local hidden-variables (LHV) theories. That is, an experiment refutes local realism if there is no LHV model compatible with their results. Bell inequalities are necessary conditions for LHV theories but, as said above, they are extremely difficult to test. The experimental difficulties with generalized Bell inequalities involving more than two particles [4] are not smaller. As a consequence there is a long history of defining restricted families of LHV theories whence easily testable inequalities may be derived. Usually the restrictions are introduced in the form of auxiliary assumptions, in addition to local realism. In the first few experiments with optical photons, performed around 1970, the auxiliary hypothesis introduced was “no enhancement”, later discussed by Clauser and Horne [5]. Similar assumptions have been used in all other experiments performed till now so that violations of the inequalities do not refute local realism but restricted families of LHV theories, namely those fulfilling the auxiliary assumptions. In my opinion the families of local hidden variables theories so far refuted by the experiments are rather unpalatable [2]. This is the case, in particular, for those fulfilling the fashionable “fair sampling hypothesis”. This is the assumption that the photons actually detected are representative of the whole set of photons emitted. But the role of hidden variables is precisely to distinguish, from each other, several physical systems in the same *pure*

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quantum state, that is systems which are identical according to the standard interpretation of quantum mechanics. Thus in any LHV model of an experiment it is natural to assume that the photons detected and those not detected correspond to different values of the hidden variables and consequently the sample of detected photons is not representative of the whole set. In conclusion, the fair sampling assumption amounts at dismissing all sensible hidden variables theories from the start.

In the present paper I study a new family of LHV models which is rather natural, in my opinion, and it allows the derivation of inequalities able to discriminate between the said family and quantum mechanics. The inequalities are easily testable in optical experiments, in particular the tests would require only moderate detection efficiencies, of the order of 30%. The said family was already considered in a previous publication [6], where an inequality fulfilled by some sub-family was proposed. That inequality has been tested empirically with the result that it was fulfilled and the quantum predictions contradicted although, according to the authors of the experiment [7], the contradiction cannot be considered a violation of quantum mechanics. In the present paper I propose inequalities which should hold true for *all members* of the said LHV family, but are contradicted by the quantum predictions in some cases.

## 2 A natural family of local hidden variables models

For the sake of clarity I shall consider experiments measuring the polarization correlation of optical photon pairs, although the generalization to other cases is possible. The set-up consists of a source of photon pairs each member of the pair travelling in a different direction, crossing a lens system, a polarization analyzer and arriving at a detector. If the polarization planes of the analyzers are determined by the angles  $\phi_1$  and  $\phi_2$ , respectively, the results of the experiment may be summarized in two single rates,  $R_1(\phi_1)$  and  $R_2(\phi_2)$ , and a coincidence rate  $R_{12}(\phi_1, \phi_2)$ . (In recent experiments four coincidence rates, rather than two, are measured because two-channel polarizers are used, a situation which will be considered below.) In a polarization correlation experiment the detection rates should be obtained from appropriately defined probabilities  $p_1, p_2$  and  $p_{12}$ , that is

$$\begin{aligned} R_j(\phi_j) &= R_0 p_j(\phi_j), \quad j = 1, 2, \\ R_{12}(\phi_1, \phi_2) &= R_0 p_{12}(\phi_1, \phi_2), \end{aligned} \quad (1)$$

where  $R_0$  is the production rate of photon pairs in the source, a quantity not measurable in standard experiments. (Here it is assumed that all photocounts come from photons produced in pairs in the source. In practice there may be counts of a different origin, e.g. dark counts in each detector, but they will be neglected for the moment, see Sect. 4 below.) Following Bell a LHV model consists of three functions,  $f(\lambda)$ ,  $Q_1(\lambda, \phi_1)$ ,  $Q_2(\lambda, \phi_2)$ , where  $\lambda$  stands

for one or several hidden variables, such that the detection probabilities could be obtained by means of the integrals

$$\begin{aligned} p_j(\phi_j) &= \int f(\lambda) Q_j(\lambda, \phi_j) d\lambda, \\ p_{12}(\phi_1, \phi_2) &= \int f(\lambda) Q_1(\lambda, \phi_1) Q_2(\lambda, \phi_2) d\lambda. \end{aligned} \quad (2)$$

The essential requirements of realism and locality imply that the said functions fulfil the conditions

$$f(\lambda) \geq 0, \quad \int f(\lambda) d\lambda = 1, \quad 0 \leq Q_j(\lambda, \phi_j) \leq 1. \quad (3)$$

A natural, but relatively simple, family of local hidden variables model is obtained if we assume that the set  $\lambda$  of hidden variables may be written  $\lambda \equiv \{\chi_1, \mu_1, \chi_2, \mu_2\}$ , where  $\chi_1$  and  $\mu_1$  ( $\chi_2$  and  $\mu_2$ ) are variables of the first (second) photon of a pair and  $\chi_j$  is a polarization angle, so that  $\chi_j$  and  $\chi_j + \pi$  represent the same polarization. Actually we may assume that  $\mu_1$  and  $\mu_2$  label a set of variables each, rather than a single one. In principle the function  $f(\cdot)$  gives the correlation amongst the four (sets of) variables. But simplifies a lot the model to assume that the variables, or sets of variables,  $\mu_1$  and  $\mu_2$  are uncorrelated amongst themselves (this is a consequence of locality) and uncorrelated with the polarization angles (this is an assumption of simplicity), so that the functions  $f(\lambda)$  and  $Q_j(\lambda, \phi_j)$  may be written

$$\begin{aligned} f(\lambda) &\equiv \rho(\chi_1, \chi_2) g_1(\mu_1) g_2(\mu_2), \\ Q_j(\lambda, \phi_j) &\equiv Q_j(\chi_j, \mu_j, \phi_j), \end{aligned} \quad (4)$$

where  $\rho$  and  $g_j$  are positive and normalized. When equations (4) are inserted in (2) and the integrals in  $\mu_j$  performed we get

$$\begin{aligned} p_j(\phi_j) &= \int \rho(\chi_1, \chi_2) P_j(\chi_j, \phi_j) d\chi_1 d\chi_2, \\ p_{12}(\phi_1, \phi_2) &= \int \rho(\chi_1, \chi_2) P_1(\chi_1, \phi_1) P_2(\chi_2, \phi_2) d\chi_1 d\chi_2, \end{aligned} \quad (5)$$

where

$$P_j(\chi_j, \phi_j) = \int g_j(\mu_j) Q_j(\chi_j, \mu_j, \phi_j) d\mu_j. \quad (6)$$

Thus our model requires defining only the functions  $\rho$  and  $P_j$ , which fulfil conditions of positivity and normalization similar to those of  $f$  and  $Q_j$ , respectively, in equations (3).

In order to derive testable inequalities from our LHV model we shall consider firstly experiments where quantum mechanics predicts rotational symmetry (in the restricted sense of invariance under rotations around an axis, see below). It is known that full rotational invariance is an additional constraint on local realism which, if fulfilled, allows deriving inequalities more stringent than the original Bell inequalities [8]. Actually quantum mechanics does not predict rotational invariance in the current two-photon experiments, because the entangled photon pairs

are produced via parametric down conversion where the optical axes of the nonlinear crystal introduce a privileged spatial orientation. In particular the predicted coincidence detection probability does not only depend on the difference  $\phi_1 - \phi_2$  but it has also a (weak) dependence on the sum  $\phi_1 + \phi_2$ . (The quantum states of the photon pairs in early experiments, involving atomic cascades, did possess rotational invariance, but they lacked good angular correlation which is a great difficulty for the discrimination between local realism and quantum mechanics [9].) In Section 4 below I shall study cases where rotational symmetry does not hold true, but firstly I will consider a symmetric model.

I shall obtain a LHV model having rotational symmetry, with respect to polarization, by using the following particular form of equations (5)

$$p_{12}(\phi) = \int \rho(\chi_1 - \chi_2)P(\chi_1 - \phi_1)P(\chi_2 - \phi_2)d\chi_1d\chi_2, \tag{7}$$

$$p_j = \int \rho(\chi_1 - \chi_2)P(\chi_j - \phi_j)d\chi_1d\chi_2, \quad j = 1, 2 \tag{8}$$

where I have also assumed symmetry amongst the polarizer-detector systems so that  $P_1 = P_2$  (models without this restriction will be discussed later on). Here and below all functions are periodic with period  $\pi$  and the integrals go from  $-\pi/2$  to  $\pi/2$ . In addition the functions  $\rho$  and  $P$  possess the following properties of positivity, symmetry and normalization ( $\rho$  is normalized so that  $p_{12}(\phi) = 1$  if  $P = 1$  in (7))

$$\rho(x) = \rho(-x) \geq 0, \tag{9}$$

$$\int \rho(x)dx = 1/\pi, \quad 0 \leq P(x) = P(-x) \leq 1.$$

I shall add the following two conditions which are plausible on physical grounds

$$\frac{d\rho(x)}{d|x|} \leq 0, \quad \frac{dP(x)}{d|x|} \leq 0. \tag{10}$$

The first inequality means that the pairs where  $\chi_1 = \chi_2$  are most likely produced in the source (this equality may correspond to parallel or perpendicular polarization, depending on the actual experiment). The second inequality means that the detection is most probable when the incoming photon has the polarization close to the plane of the analyzer. Equations (7) to (10) define our “natural” family of LHV models.

For later convenience I make the change of variables

$$\chi_1 - \phi_1 = u, \quad \chi_2 - \phi_2 = v, \quad \phi_1 - \phi_2 = \phi, \tag{11}$$

which leads to

$$p_{12}(\phi) = \int \rho(u - v + \phi)P(u)P(v)dudv. \tag{12}$$

Hence we get the following results

$$\int p_{12}(\phi)d\phi = p_1p_2, \quad p_1 = p_2 = C_0/\pi, \tag{13}$$

$$\int p_{12}(\phi) \cos 2\phi d\phi \equiv C_1^2/\pi,$$

where the constants  $C_k$  are defined by

$$C_k \equiv \int P(x) \cos(2kx)dx. \tag{14}$$

It is also convenient to introduce the new function

$$f(y) = f(-y) = \int P\left(x + \frac{y}{2}\right)P\left(x - \frac{y}{2}\right)dx \tag{15}$$

whence we get

$$p_{12}(\phi) = \int \rho(y + \phi)f(y)dy = \int \rho(y)f(y - \phi)dy, \tag{16}$$

the latter equality following from the periodicity of the functions involved.

### 3 Comparison with the quantum predictions

Now I will investigate whether the quantum predictions are compatible with the family of LHV models above defined. Quantum mechanics predicts, for experiments with rotational symmetry and all detectors having the same efficiency,  $\eta$ ,

$$p_j^Q = \frac{1}{2}\eta, \quad p_{12}^Q(\phi) = \frac{1}{4}\eta^2(1 + V \cos 2\phi). \tag{17}$$

Our family of models (7) agrees with the quantum-mechanical prediction for the single probabilities,  $p_j$ , provided we choose

$$C_0 = \int P(x)dx = \frac{\pi\eta}{2}. \tag{18}$$

Then I will search for the best LHV model of the form (7), defining “best” by the condition that, for fixed  $\eta$  and  $V$ , the prediction for the coincidence probability is as close as possible to the quantum prediction. That is the quantity  $S$  must be a minimum, where

$$S \equiv \int d\phi \left[ p_{12}(\phi) - \frac{1}{4}\eta^2(1 + V \cos 2\phi) \right]^2, \tag{19}$$

with  $p_{12}(\phi)$  given by (16) and (15) and the functions  $\rho(x)$  and  $P(x)$  fulfilling the conditions (9). For the solution of the problem I shall proceed in two steps. Firstly, for a given  $P(x)$  fulfilling (18), I search for the *best* positive and normalized function  $\rho(x)$ . In the second step I shall look for the *best*  $P(x)$ .

In the first step fixing  $P(x)$  amounts at fixing the function  $f(y)$  (see Eq. (15)). For the solution of the variational

problem, equation (19), the positivity of  $\rho$  is insured writing

$$\rho(x) = \psi(x)^2, \quad \psi \in \mathbf{R},$$

and the normalization constraint may be taken into account by means of a Lagrange parameter  $\lambda$ . Thus the problem becomes

$$\delta \left\{ S[\psi(x)^2] + \lambda \int \psi(x)^2 dx \right\} = 0.$$

Hence we get, after some algebra, that either  $\psi(x) = 0$  or

$$\int \psi(x)^2 dx \int d\phi f(x - \phi) f(y - \phi) = \lambda + \frac{1}{4} \eta^2 \int d\phi f(y - \phi) [1 + V \cos 2\phi]. \quad (20)$$

For the solution of this integral equation the properties of  $\rho(x)$ , see (7), suggest the Fourier expansion

$$\psi(x)^2 = \sum_{k=0}^{\infty} A_k \cos(2kx), \quad (21)$$

which, inserted in equation (20) leads to (see Eq. (14))

$$\sum_{k=0}^{\infty} A_k C_k^2 \cos(2ky) = \lambda + \frac{1}{4} \eta^2 [C_0^2 + V C_1^2 \cos(2y)]. \quad (22)$$

Here I have used the following relation, easily derivable from the properties of the function  $f(x)$  (see Eq. (15)),

$$\int f(u - v) \cos(2ku) du = C_k^2 \cos(2kv).$$

Equation (22) holds true for any  $y$  if

$$A_0 = \frac{4\lambda + \eta^2 C_0^2}{4C_0^4}, \quad A_1 = \frac{\eta^2 V}{4C_1^2}, \quad A_k = 0 \text{ for } k \geq 2. \quad (23)$$

Thus the function  $\rho(x)$  will be

$$\rho(x) = \frac{4\lambda + \eta^2 C_0^2}{4C_0^4} \left[ 1 + \frac{\eta^2 C_0^4 V}{C_1^2 (4\lambda + \eta^2 C_0^2)} \cos 2x \right]_+, \quad (24)$$

where  $[\cdot]_+$  means putting 0 if the quantity inside the parenthesis is negative and  $\lambda$  should be calculated from the normalization condition (see Eq. (9)). There are two cases which must be analyzed separately.

The first case corresponds to the function  $P(x)$  fulfilling

$$\frac{C_1}{C_0} = \frac{\int P(x) \cos 2x dx}{\int P(x) dx} \geq \sqrt{V}. \quad (25)$$

Then we may take  $\lambda = 0$  leading to

$$\begin{aligned} \rho(x) &= \frac{1}{\pi^2} \left[ 1 + \frac{C_0^2}{C_1^2} V \cos 2x \right] \\ &= \frac{1}{\pi^2} \left[ 1 + \frac{\pi^2 \eta^2}{4C_1^2} V \cos 2x \right], \end{aligned} \quad (26)$$

where the choice of  $C_0$ , equation (18), has been taken into account. This gives perfect agreement with the quantum predictions, (17), for both single and coincidence probabilities.

It is interesting to know the range of values of  $\eta$  and  $V$  where it is possible the agreement between the model and quantum predictions. Actually the constraints (9) and (18) put an upper bound to the left hand side of the inequality (25). It is not difficult to realize that the best function  $P(x)$ , that is the one saturating the bound, is

$$P(x) = 1 \text{ if } |x| \leq \pi\eta/4, \quad 0 \text{ otherwise.} \quad (27)$$

This choice leads to

$$C_1 = \sin(\pi\eta/2),$$

whence (25) becomes

$$V \leq V_{\max} = \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} \simeq 1 - \frac{\pi^2 \eta^2}{12}, \quad (28)$$

the latter approximation being valid for  $\eta \ll 1$ . We stress that, if this inequality is fulfilled, there are LHV models of the type (7) giving complete agreement with quantum mechanics. In fact, the choice equation (26) for  $\rho(x)$  and equation (27) for  $P(x)$  gives the desired result.

We pass to analyze the second of the two cases in equation (24) which, by the arguments of the previous paragraph, will correspond to the violation of the inequality (28). In this case the function  $\rho(z)$  may be written, correctly normalized according to equation (9),

$$\rho(z) = \frac{1}{\pi} [\pi + \tan(2\varepsilon) - 2\varepsilon]^{-1} \left( 1 + \frac{\cos 2z}{\cos 2\varepsilon} \right)_+, \quad \varepsilon \in \left( 0, \frac{\pi}{4} \right) \quad (29)$$

and I will search for the departures between the model and quantum-mechanical predictions using that function. For the single detection probability,  $p_j$ , there is no departure, that is the predictions exactly agree, provided we make the choice (18). However there is disagreement for the coincidence probability, which may be most conveniently exhibited expanding (29) in a Fourier series of the form

$$\rho(z) = \frac{1}{\pi^2} \sum_{n=0}^{\infty} a_n \cos(2nz). \quad (30)$$

We get  $a_0 = 1$ , as it should  $\rho(z)$  being normalized in the sense of (9), and

$$\begin{aligned} a_1 &= [\pi + \tan(2\varepsilon) - 2\varepsilon]^{-1} \left[ \frac{\pi - 2\varepsilon}{\cos(2\varepsilon)} + \sin(2\varepsilon) \right] \\ &= 1 + 2\varepsilon^2 - \frac{8\varepsilon^3}{\pi} + O(\varepsilon^4), \\ a_n &= \frac{2}{n(n^2 - 1)} (-1)^n \frac{\sin(2n\varepsilon) - n \tan(2\varepsilon) \cos(2n\varepsilon)}{\pi + \tan(2\varepsilon) - 2\varepsilon} \\ &= (-1)^n \frac{16}{3\pi} \varepsilon^3 + O(\varepsilon^5) \quad \text{for } n \geq 2. \end{aligned} \quad (31)$$

Hence it is straightforward to get the model predictions for  $p_{12}$  from (12 and (27), that is

$$p_{12}(\phi) = \frac{1}{4}\eta^2 \left[ 1 + \sum_{n=1}^{\infty} a_n \frac{C_n^2}{(\pi\eta/2)^2} \cos(2n\phi) \right], \quad (32)$$

where  $C_n$  was defined in equation (14).

Our aim is the find the best local model, in the sense of  $S$  (19) being a minimum for given  $\eta$  with  $P(x)$  and  $\rho(z)$  fulfilling, respectively, equations (18) and (29). It is not difficult to realize that the minimum  $S$  will correspond to  $p_{12}(\phi)$  being of the form

$$p_{12}(\phi) = \frac{1}{4}\eta^2 [1 + V \cos(2\phi) + \delta(\phi)], \quad (33)$$

with  $\delta(\phi)$  containing only terms with  $n \geq 2$  in the Fourier expansion (32). For any choice of  $P(x)$  this will be the case if the following relation between  $\varepsilon, \eta, V$  and  $C_1$  holds true

$$V = \frac{C_1^2}{(\pi\eta/2)^2} a_1. \quad (34)$$

Now we shall choose  $P(x)$  so that  $|\delta(\phi)|$  is as small as possible. From equations (31) we see that this requires that  $\varepsilon$  is small which, as  $a_1$  increases with  $\varepsilon$ , implies that  $C_1$  must be high (see Eq. (34)). Hence the best choice for  $P(x)$  will correspond to the maximum possible value of  $C_1$  compatible with equation (18), which happens if  $P(x)$  is chosen as in equation (27). After that the value of  $\varepsilon$  may be obtained from equation (34) giving, to order  $O(\varepsilon^2)$ ,

$$V = (1 + 2\varepsilon^2) \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} + O(\varepsilon^3) \\ \implies \varepsilon \simeq \sqrt{\frac{1}{2} \left( V - \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} \right)_+}, \quad (35)$$

an equality clearly showing that  $\varepsilon$  is a measure of the violation of the inequality (28) ( $\varepsilon = 0$  if the inequality holds true).

With the said choices of  $P(x)$  and  $\rho(x)$  the departure of the model from the quantum predictions is given by

$$\delta(\phi) = \sum_{n=2}^{\infty} a_n \frac{\sin^2(n\pi\eta/2)}{(n\pi\eta/2)^2} \cos(2n\phi). \quad (36)$$

Inserting here the expressions for  $a_n$ , equation (31), it is straightforward, although lengthy, to get the expression of  $\delta(\phi)$ . However the parameter  $\varepsilon$  will be very small in practice (i.e.  $\varepsilon \ll \pi/4$ ), which allows working to lowest order in  $\varepsilon$ , that is  $O(\varepsilon^3)$  (see (31)). Although that approximation of  $a_n$  is good only for the terms with small  $n$ , the terms with high  $n$  contribute but slightly. We may get the explicit form of  $\delta(\phi)$  to order  $O(\varepsilon^3)$  from equation (36) with  $a_n$  given by equation (31). We shall use the summation formula

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) = \frac{x^2}{4} - \frac{\pi^2}{12}, \quad x \in [-\pi, \pi],$$

but we must substitute  $(-1)^n \cos(n(x - \pi))$  for  $\cos(nx)$  whenever  $x > \pi$  (but  $x \leq 2\pi$ ) and a similar change if  $x < -\pi$ . In this case the summation formula needed is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) = \frac{(x - \pi)^2}{4} - \frac{\pi^2}{12}, \quad x \in [0, 2\pi].$$

Thus the term  $\delta(\phi)$  is given up to order  $O(\varepsilon^3)$  by

$$\delta(\phi) = \frac{8\varepsilon^3}{3\pi} \left[ 2 \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} \cos(2\phi) - 1 \right. \\ \left. + \frac{2}{\eta^2} \left[ \eta + \frac{2}{\pi} |\phi| - 1 \right] \Theta \left( |\phi| - \frac{\pi}{2} (1 - \eta) \right) \right], \quad (37)$$

where  $\Theta(x)$  is the Heavside function, fulfilling  $\Theta(x) = 1$  if  $x > 0$ ,  $\Theta(x) = 0$  if  $x < 0$ . (Of course we shall take  $\delta(\phi) = 0$  if Eq. (28) holds true.)

### 4 Testable inequalities for the family of local models

A parameter giving a quantitative measure of the discrepancy between quantum mechanics and the best model of the family defined by equations (7) and (8) is, from equation (36),

$$\Delta^2 = \langle \delta(\phi)^2 \rangle \equiv \frac{1}{\pi} \int \delta(\phi)^2 d\phi \\ = \frac{1}{2} \sum_{n=2}^{\infty} a_n^2 \frac{\sin^4(n\pi\eta/2)}{(n\pi\eta/2)^4}. \quad (38)$$

The summation in  $n$  would be straightforward but lengthy and, as all terms in the sum are positive, a lower bound is provided by the first few terms. With just one term we get

$$\Delta \geq \frac{\sqrt{2} \sin^3(2\varepsilon)}{3[(\pi - 2\varepsilon) \cos 2\varepsilon + \sin 2\varepsilon]} \frac{\sin^2(\pi\eta)}{\pi^2 \eta^2} \\ \simeq \frac{8\sqrt{2} \sin^2(\pi\eta)}{3\pi \pi^2 \eta^2} \varepsilon^3. \quad (39)$$

Actually it is easy to calculate  $\Delta$  to order  $O(\varepsilon^3)$  either performing the sum involved in equation (38) or directly integrating the square of the expression (37). I get

$$\Delta = \frac{4}{3\pi} \sqrt{\frac{2}{3\eta} - \frac{1}{2} - \frac{\sin^4(\pi\eta/2)}{(\pi\eta/2)^4}} \left( V - \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} \right)_+^{3/2} \\ \equiv D(\eta), \quad (40)$$

where I have taken  $\varepsilon$  from equation (35) (as before  $(.)_+$  means putting zero if the quantity inside the bracket is negative). The quantity  $D(\eta)$  gives the deviation, as defined in equation (37), between the best local model

and quantum mechanics. Thus in any experiment whose results are compatible with the family of local models defined by equations (7) and (8), the deviation will be greater than  $D(\eta)$ . That deviation  $\Delta_{\text{exp}}$  may be obtained from the measurement of the coincidence detection rates,  $R_{12}(\phi)$ , at different angles, taking  $\delta(\phi)$  to be the difference of that quantity appropriately normalized minus the best cosine fit of  $p_{12}(\phi)$ . Thus we get

$$\Delta_{\text{exp}} = \left\langle \left[ \frac{R_{12}(\phi)}{\langle R_{12}(\phi) \rangle} - 1 - V \cos 2\phi \right]^2 \right\rangle^{1/2} \geq D(\eta), \quad (41)$$

whilst quantum mechanics predicts  $\Delta_{\text{exp}} = 0$ . This inequality has the virtue that it may be tested even at relatively low detection efficiencies (provided that  $V$  is close enough to unity as is usual), so removing the biggest obstacle in the standard tests of Bell's inequalities. It replaces the previously proposed inequality (14) of reference [6], which is stronger but valid only for a family of LHV theories more restricted than (7).

From the practical point of view the inequality (41) requires the measurement of the coincidence detection rate at different values of the angle,  $\phi$ , between the polarizers. I may consider  $n$  angles defined by

$$\phi_j = \frac{j}{n}\pi, \quad j = 1, 2, \dots, n,$$

and approximate the integral in the left hand side of (41) by a sum, so leading to the inequality

$$\Delta_{\text{exp}} = \left\{ \frac{1}{n} \sum_{j=1}^n \left[ \frac{R_{12}(\phi_j)}{\langle R_{12} \rangle} - 1 - V \cos 2\phi_j \right]^2 \right\}^{1/2} \geq D(\eta), \quad (42)$$

where

$$\langle R_{12} \rangle \equiv \frac{1}{n} \sum_{j=1}^n R_{12}(\phi_j).$$

The quantity  $V$  should be taken from the quantum prediction for the actual experiment, but the inequality holds true for any value of  $V$  provided we use that same  $V$  in the calculation of  $D(\eta)$ , equation (41). It is interesting to use the value of  $V$  giving the minimum value to  $\Delta_{\text{exp}}$  for the given empirical rates  $\{R_{12}(\phi_j)\}$ , that is

$$V = 2 \frac{\sum_{j=1}^n R_{12}(\phi_j) \cos 2\phi_j}{\sum_{j=1}^n R_{12}(\phi_j)}, \quad (43)$$

where I have taken into account the equalities

$$\sum_{j=1}^n \cos 2\phi_j = 0, \quad \sum_{j=1}^n \cos^2 2\phi_j = \frac{n}{2}.$$

In this case I get

$$\Delta_{\text{min}} =$$

$$\left\{ \frac{n \sum_{j=1}^n R_{12}^2(\phi_j) - 2(\sum_{j=1}^n R_{12}(\phi_j) \cos 2\phi_j)^2}{(\sum_{j=1}^n R_{12}(\phi_j))^2} - 1 \right\}^{1/2} \geq D(\eta), \quad (44)$$

this inequality making possible empirical tests of the family of local models defined by equations (7) and (8) without any reference to quantum mechanics, whilst equation (42) allows tests of the local models versus quantum mechanics.

The test of the inequalities involve the measurement of  $n/2 + 1$  rates if  $n$  is even or  $(n+1)/2$  rates if  $n$  is odd, once we take into account the symmetry  $R_{12}(\phi_j) = R_{12}(\pi - \phi_j)$ . For even  $n \geq 4$ , equation (44) may be related to the Fourier expansion equation (38) using the interpolation formula

$$\frac{R_{12}(\phi_j)}{\langle R_{12} \rangle} = 1 + \sum_{k=1}^{n/2} b_k \cos(2k\phi_j), \quad (45)$$

whence the left hand side of (44) may be interpreted as

$$\Delta_{\text{min}} = \left\{ \frac{1}{2} \sum_{k=2}^{n/2-1} b_k^2 + b_{n/2}^2 \right\}^{1/2}. \quad (46)$$

The most simple non-trivial case corresponds to  $n = 4$ , that is the choice of the angles  $0, \pi/4$  and  $\pi/2$ . In this case equations (41) and (44) lead to the inequality

$$\Delta_{\text{min}} = \frac{R_{12}(0) + R_{12}(\frac{\pi}{2}) - 2R_{12}(\frac{\pi}{4})}{R_{12}(0) + R_{12}(\frac{\pi}{2}) + 2R_{12}(\frac{\pi}{4})} \gtrsim D(\eta). \quad (47)$$

Strictly speaking, from equation (44) we may only conclude that the modulus of the left hand side fulfils the inequality, but we may take that side as positive because  $a_2 > 0$  in the expansion (30). At a difference with (41) the inequality (47) is not rigorous due to the approximation made in going from equation (41) to (42), the possible error being greater as the number of measured rates is smaller.

In addition to the inequalities it is possible to derive some predictions of the proposed local models for measurable quantities. For instance, defining

$$V_A \equiv \frac{R_{12}(0) - R_{12}(\frac{\pi}{2})}{R_{12}(0) + R_{12}(\frac{\pi}{2})},$$

$$V_B \equiv \sqrt{2} \frac{R_{12}(\frac{\pi}{8}) - R_{12}(\frac{3\pi}{8})}{R_{12}(\frac{\pi}{8}) + R_{12}(\frac{3\pi}{8})}, \quad (48)$$

their difference may be estimated. The former quantity is called the visibility (or contrast) of the polarization correlation curve and the latter is the quantity measured in the standard tests of Bell's inequalities. According to quantum mechanics both quantities should be equal to the parameter  $V$ , (17), but in our model there is a difference. In fact, from equation (45) it follows that

$$V_A = \frac{b_1 + b_3}{1 + b_2 + b_4} \simeq b_1 + b_3 - b_2 - b_4,$$

$$V_B = \frac{b_1 - b_3}{1 - b_4} \simeq b_1 - b_3 + b_4,$$

where I have taken into account that, from a comparison of the empirical quantities  $b_k$  with the model quantities of equations (36) and (35), we get

$$b_4 \lesssim |b_3| \lesssim b_2 \ll 1 \simeq b_1 \equiv V.$$

Hence the family of LHV models here studied predicts (see Eqs. (31) and (35))

$$V_B - V_A \simeq b_2 - 2b_3 + 2b_4 \simeq \frac{20\sqrt{2}}{3\pi} K \left( V - \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} \right)_+^{3/2}, \quad (49)$$

$K$  being a number of order unity. In some experiments it is reported, instead of  $V_A$ , the value of  $V$ , which is obtained from a fit to the empirical coincidence rates  $R_{12}(\phi_j)$ . In this case the model predicts

$$V_B - V \simeq -b_3 \simeq \frac{4\sqrt{2}}{3\pi} K \left( V - \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} \right)_+^{3/2}. \quad (50)$$

I stress that these estimates are correct only for the “best” local model of the family. Thus their value is only indicative of the order of magnitude to be expected in actual experiments if they are compatible with the said family of local models.

In recent optical tests of Bell’s inequalities people use two-channel polarizers and four coincidence rates are measured,  $p_{++}(\phi)$ ,  $p_{+-}(\phi)$ ,  $p_{-+}(\phi)$ ,  $p_{--}(\phi)$ . Typically there is symmetry between the two channels, at least approximate. In this case our model may be extended to these experiments using two functions  $P_+(\phi)$  and  $P_-(\phi) = P_+(\phi + \pi/2)$  instead of only one,  $P(\phi)$ , as in equation (7). Assuming the form (27) for  $P_+(\phi)$  we get the prediction

$$R_{+-}(\phi + \pi/2) = R_{-+}(\phi + \pi/2) = R_{++}(\phi) = R_{--}(\phi), \quad (51)$$

the latter given by equation (33) (putting  $\varepsilon = 0$  if the inequality (28) holds true). The quantity reported in the experiments is the correlation, defined by

$$E(\phi) = \frac{R_{++}(\phi) + R_{--}(\phi) - R_{+-}(\phi) - R_{-+}(\phi)}{R_{++}(\phi) + R_{--}(\phi) + R_{+-}(\phi) + R_{-+}(\phi)}, \quad (52)$$

for which our model predicts

$$E(\phi) = V \cos(2\phi) - \delta(\pi/2 + \phi), \quad (53)$$

where  $\delta(\phi)$  is given by equation (37). The parameter most accurately measured in the experiments is

$$S = |3E(\pi/8) - E(3\pi/8)| \equiv 2\sqrt{2}V_B, \quad (54)$$

which is predicted to have the value  $2\sqrt{2}$  according to quantum mechanics in ideal experiments with 100% detection efficiency, but  $S$  should be less than 2 for any LHV model fulfilling the “fair sampling hypothesis” (see Introduction section).

From equation (48) it is easy to see that, in our LHV family of models, the quantities  $V_B$  and  $V_A$  defined in (48) correspond, respectively, to  $2\sqrt{2}$  times the quantity  $S$  of equation (52) and to

$$V_A = \frac{1}{2}(E(0) - E(\pi/2)). \quad (55)$$

This allows testing the model prediction equation (49) in experiments involving two-channel polarizers from measurements of the correlation  $E(\phi)$ , equation (52), at the four angles  $0, \pi/8, 3\pi/8$  and  $\pi/2$ . Also the prediction equation (50) may be tested getting  $V_B$  from (54) and  $V$  from a fit of  $E(\phi)$  as shown in equation (53).

## 5 From ideal to real experiments

In previous sections, I have analyzed rather idealized experiments. In actual polarization correlation experiments involving parametric down converted photons the results obtained are not so simple as assumed in equations (1) and (17). Indeed it is frequent that the coincidence detection rate is not rotationally invariant in the sense that  $R_{12}(\phi_1, \phi_2)$  does depends on the angles  $\phi_1$  and  $\phi_2$  separately rather than on the difference  $\phi = \phi_1 - \phi_2$  only. In addition, the single rates may depend on the positions of the polarizers, that is  $R_j = R_j(\phi_j)$ , and the efficiencies,  $\eta_1$  and  $\eta_2$ , of the detectors may not be the same. Finally not all photons may arrive in pairs at the detectors (either because there is some single photon production or because a fraction of the photons do not enter the apertures.) The relevance of these nonidealities will be analyzed in present section.

The quantities which may be measured in typical experiments involving single channel polarizers are: two single rates,  $R_1$  and  $R_2$ , the rate  $R_{12}$  of counts in the second detector conditional to counts in the first detector (within an appropriate time window) and the quantum efficiencies of the detectors (these measured in auxiliary experiments). In order to apply the analysis of Bell [1] to the experiments we should define single and coincidence probabilities. But for the test of the family of LHV theories studied in this paper (defined by Eqs. (7) and (8)) we do not need the single probabilities and the single rates,  $R_j$ , are not used. In order to obtain the coincidence probability,  $p_{12}$ , I introduce the rate,  $R_0$ , of pair production in the source, whence I may get the  $p_{12}$  using Bayes rule of probability theory as follows

$$p_{12} = p_1 \times p_{2/1} = \frac{R_1}{R_0} \times \frac{R_{12}}{R_1} = \frac{R_{12}}{R_0}. \quad (56)$$

The adequacy of this definition of  $p_{12}$  requires that all photons leaving the source are produced in pairs, with no more than one pair within each time-window. That is, equation (56) amounts at neglecting single photon production, dark counts and the possible production of several photon pairs within one time-window. I shall assume that all these difficulties are avoided by a subtraction of

“accidental coincidences”, which is a standard practice in polarization correlation experiments. (I stress that this is so because we are attempting to test a restricted family of LHV models, given by Eqs. (5), but would not be valid if we attempted to refute the whole family of LHV theories [2].) Actually the pair production rate  $R_0$  cannot be measured in the experiments but this is not a real difficulty. Indeed the inequalities fulfilled by the LHV family of models proposed in this paper involve only the dependence of the coincidence probability on the positions of the polarizers, determined by the angles  $\phi_1$  and  $\phi_2$ . In the experiments this dependence is usually studied by measurements with a polarizer at a fixed position, say  $\phi_2$  fixed, and the other polarizer’s position varied. Typically the measurements are made at two different positions of the second polarizer, say  $\phi_2 = 0$  and  $\phi_2 = \pi/8$ . An appropriate interpolation leads to the following parametrization

$$p_{12} \propto R_{12} \propto 1 + [V_m + (V_M - V_m) \cos 4\phi_2] \cos 2\phi, \quad (57)$$

where  $V_M(V_m)$  is the maximum (minimum) visibility of the detection coincidence curve when the angle  $\phi_2$  is varied.

A local hidden-variables model able to reproduce equation (57) for not too high detection efficiencies is given by the functions  $\rho$  and  $P_j$  (see Eq. (5))

$$\rho(\chi_1, \chi_2) = \frac{1}{\pi^2} \{1 + [W_m + (W_M - W_m) \cos 4\chi_2] \cos 2\chi\},$$

$$P_j(\chi_j, \phi_j) = \beta_j \Theta \left( \frac{\pi \eta_j}{4\beta_j} - |\chi_j - \phi_j| \right), \quad j = 1, 2, \quad (58)$$

with any  $\beta_j \in [\eta_j, 1]$  and appropriate choices for  $W_M$  and  $W_m \leq W_M$ . Here  $\chi = \chi_1 - \chi_2$  and  $\Theta(\cdot)$  is the Heaviside function  $\Theta(x) = 1$  if  $x \geq 0$ , zero otherwise.  $\eta_1(\eta_2)$  is assumed to be the efficiency of all detectors of the first, say Alice’s (second, say Bob’s) photons. If the detection efficiencies are high enough and the value of  $V_M$  in (57) is close enough to unity, then it is not possible to reproduce the quantum predictions. In fact, the conditions (9) imply  $W_M \leq 1$  (in addition to  $\beta_j \leq 1$ ) which give after some algebra the constraint

$$V_M \leq \frac{1}{3}(V_M + V_m)(s_1^2 + s_2^2) + \frac{4}{\pi^2} \frac{s_1 s_2}{\eta_1 \eta_2} \left[ 1 - \frac{2}{3}(s_1^2 + s_2^2) \right],$$

$$s_j \equiv \sin \left( \frac{\pi \eta_j}{2} \right). \quad (59)$$

If  $\eta_j \ll 1$  and both  $V_M$  and  $V_m$  are close to unity, as is usually the case in typical experiments (where  $V_m \gtrsim 0.96, \eta_j \lesssim 0.3$ ), the inequality becomes (compare with (28))

$$V_M \lesssim 1 - \frac{\pi^2}{24}(\eta_1^2 + \eta_2^2). \quad (60)$$

The question whether the quantities  $\eta_j$  refer here to the quantum efficiencies of the photon detectors (photodiodes) or to the overall detection efficiency (which may be substantially smaller than  $\eta_j$  due to several losses) is discussed in next section. Agreement between the model and quantum predictions is possible only if equation (59)

(or Eq. (60)) is fulfilled. If this is not the case, the model predicts inequalities which are violated by quantum mechanics. In the following I study this case, which allows empirical discrimination of the family of local models here proposed versus quantum mechanics.

From the results of Section 3 we see that the best choice for the functions  $P_j$  is provided by equation (27) (or, what is the same, Eq. (58) with  $\beta_j = 1$ ). “Best” is defined in the sense that the disagreement with the quantum prediction is a minimum, the disagreement being measured by an appropriate generalization of equation (19). In order to get the best function  $\rho$  we may assume that the empirical results to be compared with the LHV model prediction are obtained with a fixed  $\phi_2$  (see Eq. (57)). In this case all inequalities of Section 4 are valid with the effective visibility,  $V_{eff}$ , of the correlation curve substituted for  $V$ , where we define

$$V_{eff} \equiv [V_m + (V_M - V_m) \cos 4\phi_2].$$

It is easy to see that the stringent inequalities are obtained when  $V_{eff} \equiv V_M$ , corresponding to  $\phi_2 = 0$ . Thus the inequalities derived in previous section apply, with  $V_M$  substituted for  $V$ , including the extension to experiments using two-channel polarizers and four detectors (a trivial change is required if the efficiencies of the detectors are not the same).

## 6 Discussion

Empirical tests of the models defined by equations (7) to (10) are possible by means of the inequalities (28) and (42), any experiment where one of these inequalities holds true being compatible with a local hidden variables model of the family. A relevant question is the choice of the efficiency,  $\eta$ , which should enter in these inequalities. For the family of LHV models defined by equations (7) to (10), a comparison with the quantum prediction equation (17) shows that  $\eta$  corresponds to twice the ratio between the coincidence rate averaged over angles and the single rate, that is

$$\eta_1 = 2 \frac{\langle R_{12}(\phi) \rangle}{R_2}, \quad \eta_2 = 2 \frac{\langle R_{12}(\phi) \rangle}{R_1} \quad (61)$$

where equation (1) is used and I have made a trivial generalization allowing for the case  $R_1 \neq R_2$ . This definition of  $\eta_j$  corresponds to the overall detection efficiency, which takes into account all kinds of losses in lenses, filters and polarizers as well as the fact that the angular correlation, although good in parametric down converted photon pairs, is not perfect. The value of  $\eta_j$  so defined is substantially smaller than the quantum efficiency of the corresponding detector. With this choice of  $\eta_j$  the violation of the inequalities (59) or (60) would be difficult in actual experiments, thus preventing the discrimination of the family of LHV models against quantum mechanics. For this reason I propose to restrict the family of LHV models by adding, to equations (7) to (10), the auxiliary assumption that all losses above mentioned may be taken into account by



some constants  $K_j < 0$ , the same for single and coincidence rates, which represent the fraction of the photons produced in the source that arrive at the detectors, that is assuming the following

$$R_j = \frac{1}{2}R_0K_j\eta_j, \quad \langle R_{12}(\phi) \rangle = \frac{1}{4}R_0K_1K_2\eta_1\eta_2, \quad (62)$$

where now  $\eta_1$  and  $\eta_2$  are the quantum efficiencies of the two photon detectors (if there are four, as is usual, we assume that the two detectors on each side have the same efficiency; if this were not the case we should use an appropriate average). Equations (62) represent a kind of “fair sampling” applied only to lenses, filters and polarizers (but not to the detectors). In my opinion this restricted fair sampling assumption is far more plausible than the standard one, which I strongly criticized in Introduction section. In order to construct a LHV model consistent with equations (62) we should replace equations (7) and (8) by

$$p_{12}(\phi) = K_1K_2 \int \rho(\chi_1 - \chi_2)P_1(\chi_1 - \phi_1) \times P_2(\chi_2 - \phi_2) d\chi_1 d\chi_2, \quad (63)$$

$$p_j(\phi) = K_j \int \rho(\chi_1 - \chi_2)P_j(\chi_1 - \phi_1) d\chi_1 d\chi_2. \quad (64)$$

All results of the present paper will remain valid, in particular the inequalities (28) and (42), except that now  $\eta_j$  will be the efficiencies of the photon detectors rather than the overall detection efficiencies, including all kinds of losses.

With the choice of  $\eta_j$  proposed in the previous paragraph the tests of the proposed inequalities are not difficult because visibilities about 97% and detection efficiencies of the order of 20% would be enough and these conditions have already been achieved in performed experiments. For instance the experiment by Kurtsiefer et al. [10] reports  $V_M = 0.982 \pm 0.001$  and  $V_m = 0.970 \pm 0.001$  after an appropriate subtraction of accidentals. Any of these values, combined with the efficiency  $\eta = 0.214$ , leads to a violation of the inequality (28), thus making the test possible using the inequalities (42) or (47). The quantities reported in the published paper do not allow the test but the experiment clearly shows that the empirical tests may be easily performed with present technology.

The discrimination between the family of local models and quantum mechanics is substantially more difficult than just to test the models. In fact, calculating the predictions of quantum mechanics in actual (non-ideal) experiments is far from trivial. For instance, in the ideal case the quantum prediction is given by equation (17) with both the detection efficiency and the visibility of the coincidence curve equal to unity, i.e.  $V = \eta = 1$ . With these values both inequalities (28) and (42) are violated, which excludes all local models of the family here studied. This

agrees with the well-known fact that no local hidden variables model is compatible with quantum mechanics for all (ideal) experiments. Nevertheless the non-perfect behaviour of detectors lowers their efficiency,  $\eta$ , making the results compatible with local models in all experiments performed till now, a fact which should be well-known (but is not so [2]).

In summary, the family of local models studied in this paper may be empirically refuted with a moderate value of the detection efficiency,  $\eta$ , combined with a relatively high value of the visibility,  $V$ , provided that there is rotational symmetry, as assumed in Section 3. However in the fashionable parametric down conversion experiments rotational symmetry does not hold true, which makes easier the construction of local models compatible with the experiments. In particular it is not obvious that quantum mechanics, with all non-idealities taken into account, predicts a detection counting rate of the form (57). If this is not the case the fulfillment of the inequalities (42) or (47) would not imply a violation of quantum predictions. That is the non-ideal behaviour of any actual experiment increases substantially the range of parameters where quantum mechanics is compatible with local hidden variables theories, thus making the empirical discrimination rather difficult. It may even be the case that no actual, i.e. non-ideal, experiment allows discriminating local hidden variables versus quantum mechanics.

I thank Marco Genovese for useful comments and an anonymous referee for several suggestion which improved the clarity of the paper.

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